Lecture 30

Sunday, March 20, 2022 1:43 AM

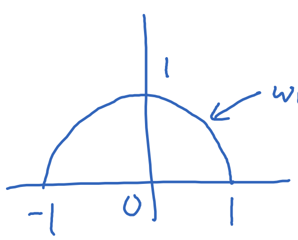
* Prayer

* Spiritual thought

Integral in calc I : $\int_{[a,b]} f(x)\,dx$

Multivariable calc : $\underbrace{\iint_D f(x,y)\,dA}_{\text{volume}}$, $\underbrace{\iiint_E f(x,y,z)\,dV}_{\text{mass / volume}}$

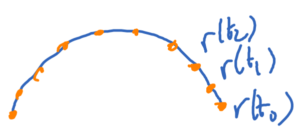
Consider the following problem:



wire with mass density (per unit length) : $f(x,y) = x^2$

What is the mass of the wire?

parametrization : $r(t) : \begin{cases} x = \cos t \\ y = \sin t \end{cases}$ $\quad 0 \le t \le \pi$



The length of the arc from $r(t_k)$ to $r(t_{k+1})$ is approximately $|r(t_{k+1}) - r(t_k)| \approx |r'(t_k)|\,\Delta t$

The mass of this arc is approximately

$$\Delta m = |r'(t_k)|\,\Delta t \; f(x(t_k), y(t_k)).$$

The total mass is approximately

$$m = \sum \Delta m \approx \sum f(x(t_k), y(t_k)) \, |r'(t_k)| \, \Delta t$$

To be precise,

$$m = \int_0^{\pi} f(x(t), y(t)) \underbrace{|r'(t)| \, dt}_{ds} = \int_C f(x,y) \, ds$$

This is called the line integral of f over the curve C.

In general, $C: \; r(t) = (x(t), y(t), \dots) \qquad a \le t \le b$

$$f = f(x, y, \dots)$$

$$\int_C f(x,y,\dots) \, ds \;\underline{\underline{\text{def}}}\; \int_a^b f(x(t), y(t), \dots) \underbrace{|r'(t)|}_{\sqrt{x'(t)^2 + y'(t)^2 + \dots}} \, dt$$

$\underline{\underline{\text{Ex}}}$

wire $C: \; r(t) = (\cos t, \sin t, t), \quad 0 \le t \le 2\pi$

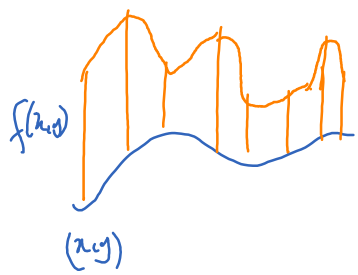
$f(x,y,z) = z$ (the higher altitude the heavier the wire is)

what is the mass of the wire?

$$\int_C f(x,y,z) \, ds = \int_C z \, ds = \int_0^{2\pi} t \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt$$

$$= \int_0^{2\pi} t \underbrace{\sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}}_{=\sqrt{2}} \, dt = 2\sqrt{2}\pi^2$$

Alternative interpretation: area of the wall built on the curve



Mathematica:

```
x = Cos[t]
y = Sin[t]
z = t
r = {x, y, z}
ds = Sqrt[D[r, t] . D[r, t]]
Integrate[z * ds, {t, 0, 2π}]
```